Formal Characters.

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Notation: Λ: weight lattice ; Δ: root lattice ; $\Gamma = \mathbb{Z}^+\Delta$.

Definition of characters.

Idea: character of a rep determines it uniquely up to equivalence.

(But it is possible only for finite dim)

Step 1. finite dim rep — $\mathbb{Z}\Lambda$ (ring)

To avoid confusion, associate each $\lambda \in \Lambda$ a symbol $e(\lambda)$.

$$\mathrm{ch}\, M := \sum_{\lambda\in\Lambda} \dim M_\lambda \cdot e(\lambda)$$

- $\mathrm{ch}(M \oplus N) = \mathrm{ch}\, M + \mathrm{ch}\, N$
- $\mathrm{ch}(M \otimes N) = \mathrm{ch}\, M \cdot \mathrm{ch}\, N$

By Weyl's complete thm, it is enough to know $\mathrm{ch}\, L(\lambda)$.

Step 2. Modules in $\mathcal{O}$ — $\mathcal{X}$

$$\mathcal{X} := \{ f : \mathfrak{h}^* \to \mathbb{Z} \ : \ \mathrm{Supp}(f) \subseteq \bigcup_{\text{fin } \lambda} (\lambda - \Gamma) \}$$

convolution product: $(f * g)(\lambda) := \sum_{\nu+\mu=\lambda} f(\mu)\cdot g(\nu)$

- $\mathcal{X}$ is a commutative ring under convolution.
- $(\mathrm{ch}\, M)(\lambda) = \dim M_\lambda$, $e_\lambda(\mu) = \delta_{\lambda\mu}$
- $\mathcal{X}_0$ the additive group of $\mathcal{X}$ gen by all $\mathrm{ch}\, M$.

Prop 1) $0 \to M' \to M \to M'' \to 0$ SES in $\mathcal{O}$, we have $\mathrm{ch}\, M' + \mathrm{ch}\, M'' = \mathrm{ch}\, M$.

2) $\mathcal{X}_0 \xrightarrow{\sim} K(\mathcal{O})$; $\mathrm{ch}\, M \to [M]$

3) If $M \in \mathcal{O}$ and $\dim L < \infty$, $\mathrm{ch}(M \otimes L) = \mathrm{ch}\, M * \mathrm{ch}\, L$

4) If M finite dim, $w \in W$ Weyl group, $w \cdot \mathrm{ch}\, M := \sum \dim M_\lambda\, e(w\lambda) = \mathrm{ch}\, M$.
i.e. $\mathrm{ch}\, M$ is W-invarient.

Q: What is the characters of Verma module $M(\lambda)$

$$\mathrm{ch}\, M(0)(\gamma) = \#\{ (c_\alpha)_{\alpha\in\Phi^+} \mid c_\alpha \geq 0,\ \gamma = \sum_{\alpha\in\Phi^+} c_\alpha \cdot \alpha \} := p(\gamma) \quad \text{Kostant number}$$

Prop For any $\lambda \in \mathfrak{h}^*$, $ch M(\lambda) = ch M(0) * e(\lambda) = p * e(\lambda)$

Pf: $\dim M(\lambda)_\nu = p(\nu - \lambda)$

Ex. $ch L(\lambda)$ with $\lambda \in \mathfrak{h}^*$ are linearly independent in $\mathcal{X}$ and form a basis of $\mathcal{X}_0$.

If $\sum k_\lambda ch L(\lambda) = 0$, then consider the maximal weight γ among nonzero k_λ.

Then $[\sum k_\lambda ch L(\lambda)](\gamma) = k(\gamma) = 0$ which is a conflict. $\Rightarrow$ linearly independent.

Q: What is the characters of $L(\lambda)$. (not easy !!) only consider $\lambda \in \Lambda^+$.

- $ch M(\lambda) = \sum_{\substack{\mu \leq \lambda \\ \mu \in W\cdot\lambda}} a(\lambda,\mu)\, ch L(\lambda) = \sum_{w\cdot\lambda \leq \lambda} a(\lambda, w)\, ch L(w\cdot\lambda)$

where $a(\lambda,\mu) = [M(\lambda) : L(\mu)] \geq 0$ and $a(\lambda,\lambda) = 1$

- $ch L(\lambda) = \sum b(\lambda, w)\, ch M(w\cdot\lambda)$ where $b(\lambda,w) \in \mathbb{Z}$ and $b(\lambda,1) = 1$

The function p & q

Recall: $p = ch M(0)$, $ch M(\lambda) = p * e(\lambda)$

Define $f_\alpha(\lambda) := \begin{cases} 1 & \text{if } \lambda = -k\alpha \text{ for some } k \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases}$

$$f_\alpha = e(0) + e(-\alpha) + e(-2\alpha) + \cdots$$

Lemma A: a) $p = \prod_{\alpha>0} f_\alpha$ ($\alpha \in \Phi^+$)   b) $(e(0) - e(-\alpha)) * f_\alpha = e(0)$

Pf. a) $\prod_{\alpha>0} f_\alpha = \sum_{(c_\beta) \in \mathbb{Z}_{\geq 0}^{|\Phi^+|}} \prod_{\alpha \in \Phi^+} e(-c_\alpha \cdot \alpha) = p$

Define $q := \prod_{\alpha>0} \left(e(\frac{\alpha}{2}) - e(-\frac{\alpha}{2})\right)$, then $q = \prod_{\alpha>0} e(\frac{\alpha}{2}) * (e(0) - e(-\alpha)) = e(\rho) * \prod_{\alpha>0} (e(0) - e(-\alpha))$

Note that $q \neq 0$, because $q(\rho) = 1$.

Lemma B. For all $w \in W$, we have $wq = (-1)^{\ell(w)} q$

Pf. If $w = 1$, there is nothing to prove. If $w = s_\alpha$, w sends α to $-\alpha$ but keeps all other positive roots in $\Phi^+ \Rightarrow w q = -q$. It is enough to show on simple reflections.

<u>Lemma C</u>. For each $\lambda \in \mathfrak{h}^*$, $q * \mathrm{ch}\, M(\lambda) = q * p * e(\lambda) = e(\lambda+\rho)$

pf.
$$q * p = e(\rho) * \prod_{\alpha>0} \left(e(0) - e(-\alpha)\right) * \prod_{\beta>0} f_\beta$$
$$= e(\rho) * \prod_{\alpha>0} \left(e(0) - e(-\alpha)\right) * f_\alpha$$
$$= e(\rho)$$

Formulas of Weyl and Kostant.

<u>Thm</u>. (Weyl) Let $\lambda \in \Lambda^+$ ($\dim L(\lambda) < \infty$), Then
$$q * \mathrm{ch}\, L(\lambda) = \sum_{w \in W} (-1)^{l(w)} e(w(\lambda+\rho))$$
In particular, when $\lambda = 0$, $q = \sum_{w \in W} (-1)^{l(w)} e(w\rho)$

pf. $\mathrm{ch}\, L(\lambda) = \sum_{w\cdot\lambda \leq \lambda} b(\lambda, w)\, \mathrm{ch}\, M(w\cdot\lambda) = \sum_{w \in W} b(\lambda, w)\, p * e(w\cdot\lambda)$

Multiply both sides by q:
$$q * \mathrm{ch}\, L(\lambda) = \sum b(\lambda,w)\, q * p * e(w\cdot\lambda)$$
$$= \sum b(\lambda,w)\, e(\rho) * e(w\cdot\lambda)$$
$$= \sum b(\lambda,w)\, e(w\cdot\lambda+\rho)$$
$$= \sum b(\lambda,w)\, e(w(\lambda+\rho))$$

Apply s_α to both sides:

$$s_\alpha(q * \mathrm{ch}\, L(\lambda)) = s_\alpha q * \underbrace{s_\alpha \mathrm{ch}\, L(\lambda)}_{\text{ch of f.dim is W-invariant}} = -q * \mathrm{ch}\, L(\lambda)$$
$$s_\alpha e(w(\lambda+\rho)) = e(s_\alpha w(\lambda+\rho))$$
$$\Rightarrow b(\lambda, w) = -b(\lambda, s_\alpha w)$$

Induction on the length of w, we have $b(\lambda,w) = (-1)^{l(w)} \underbrace{b(\lambda, 1)}_{b(\lambda,1)=1} = (-1)^{l(w)}$

<u>Cor</u> (Kostant). If $\lambda \in \Lambda^+$ and $\mu \leq \lambda$, then
$$\dim L(\lambda)_\mu = \sum_{w \in W} (-1)^{l(w)} p(\mu - w\cdot\lambda) = \sum_{w \in W} (-1)^{l(w)} p((\mu+\rho) - w(\lambda+\rho))$$

pf. $\mathrm{ch}\, L(\lambda) = q * p * e(-\rho) * \mathrm{ch}\, L(\lambda) = p * e(-\rho) * \sum (-1)^{l(w)} e(w\cdot\lambda+\rho)$
$$= p * \sum (-1)^{l(w)} e(w\cdot\lambda)$$